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SERIES SOLUTION FOR HEAT TRANSFER THROUGH A TURBULENT BOUNDARY LAYER

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Abstract—A series solution has been found to the parabolic partial differential equation governing heat transfer from a smooth wall into a steady, uniform-property turbulent boundary layer. The solution is valid for the initial portion of the wall immediately downstream of a step-wise discontinuity in wall temperature. The differential equation represents the temperature distribution in a fluid, with Prandtl number greater than 0·5, when a universal velocity profile exists in the fluid wherever the temperature gradient is appreciable. To obtain a series solution to the differential equation, the diffusivities of heat and momentum were each expressed in the form of a power series in u^+. A general form was used for the power series which can be easily modified to fit many of the diffusivity profiles recommended in the literature. As an example of this flexibility, computations were made (for several Prandtl numbers) for three commonly-used diffusivity profiles. Some of the coefficients of u^+ in the diffusivity power series have been deduced by comparison of the solutions of the differential equation with recently published experimental data.

NOMENCLATURE

a_b, a constant (equation 2);
A, a constant $[\equiv N_{Pr}/N_{Pr,t}]$ (equation 17);
b, a constant (equation 2);
B, a constant (equation 25);
c_f, local drag coefficient $[\equiv 2\tau_S/(\rho\mu_G^2)]$ (equation 10);
C, a constant (equation 25);
E, a constant (equation 4);
F_n, a constant (equation 15);
k, a constant (equation 2);
K, a constant (equation 4);
L, length of heat- (or mass-) transfer section;
L^+, non-dimensional form of L (equation 37);
N_{Pr}, laminar Prandtl number (equation 3);
$N_{Re,d}$, Reynolds number based on pipe diameter;
$N_{Pr,t}$, turbulent Prandtl number (equation 3);
N_{Sc}, laminar Schmidt number;
N_{St}, Stanton number (equation 10);
S, $\equiv N_{St}\, N_{Pr}/\sqrt{(c_f/2)}$ (equation 10);
$\bar{S}$, defined by equation (37);
t, temperature [°C];
u, velocity in x-direction [m/h];
u^+, $\equiv u/\tau_S/\rho)^{\frac{1}{2}}$ (equation 1);
x, distance along the wall from the leading edge [m];
x^+, non-dimensional distance along the wall $[\equiv \int_0^x \sqrt{(c_f/2)}(\rho u_G/\mu)\,\mathrm{d}x]$ (equation 1);
y, perpendicular distance from wall [m];
y^+, $\equiv y(\tau_S\rho)^{\frac{1}{2}}/\mu$ (equation 6);
Z, $\equiv x^+/N_{Pr}$ (equation 18).

Greek symbols

ε_h^+, non-dimensional effective conductivity (equation 1);
ε_u^+, non-dimensional effective viscosity (equation 1);
γ, $\equiv a_b^{1/b}(u^+)$ (equation 14);
θ, $\equiv \dfrac{t - t_S}{t_G - t_S}$ (equation 1);
μ, molecular viscosity [kg/mh] (equation 21);
ν, $\equiv \gamma Z^{-1/3}$ (equation 21);
ρ, density [kg/m³];

τ, shear stress in boundary layer [kg/mh^2];

φ, dimensionless form of temperature (equation I-10).

Subscripts

G, refers to conditions in bulk of fluid stream;

S, refers to conditions in fluid immediately at the wall.

1. INTRODUCTION

1.1 *The Problem*

THE PRIMARY purpose of this paper is to develop an exact analytic expression for the heat transfer from an isothermal spanwise strip on a smooth wall to a steady, uniform-property stream; a universal velocity profile is assumed to exist everywhere in the fluid where the temperature gradient is appreciable. A series method of solution is used. The expression obtained by this method for the local heat transfer depends on the relationships which are postulated between the velocity, and the eddy diffusivities of heat and momentum. Use of specific forms of these relationships in the analysis, followed by comparisons of the final solutions with experimental data, permits information about the correct forms of the diffusivity relationships to be obtained.

1.2 *Description of Basic Equations used in the Analysis*

1.2.1 *The parabolic partial differential equation*

The parabolic partial differential equation governing the temperature near the wall in a turbulent boundary layer was shown in reference [1] to be:

$$\frac{\partial \theta}{\partial x^+} = \frac{1}{u^+ \varepsilon_u^+} \frac{\partial}{\partial u^+}\left\{\frac{\varepsilon_h^+}{\varepsilon_u^+}\frac{\partial \theta}{\partial u^+}\right\} \quad (1)$$

where u^+, x^+, ε_u^+, ε_h^+, and θ are respectively the dimensionless measures of: velocity, distance along the wall, effective viscosity, effective conductivity, and temperature. In developing equation (1), it was necessary to assume that a universal velocity profile existed at all points in the fluid where the temperature gradient was appreciable.

1.2.2 *The diffusivity profiles*

1.2.2.1 *General form.* For the method of solution of equation (1) used in this paper, it is necessary to assume that the diffusivities of momentum and heat can be expressed as a power series in u^+. Specifically,

$$\varepsilon_u^+ = 1 + a_b (u^+)^b + \ldots + a_{b+k} (u^+)^{b+k} + \ldots \quad (2)$$

$$\varepsilon_h^+ = \frac{1}{N_{Pr}} + \frac{1}{N_{Pr,t}}\{\varepsilon_u^+ - 1\} \quad (3)$$

where the a_{b+k} terms are constants, any of which may be zero.

Hinze [2] reported that Reichardt [3] and Elrod [4] showed there are theoretical reasons against b in equation (2) being less than three if the shear stress varies along the wall, and four if there is no such variation. For this reason the minimum value of b used in this study will be three.

1.2.2.2 *Spalding's diffusivity profiles.* Two eddy diffusivity profiles were suggested in reference [5], one for a value of b in equation (2) of 3, and the other for $b = 4$. For $b = 4$, the recommended profile was

$$\varepsilon_u^+ = 1 + \frac{K}{E}\left\{\frac{(Ku^+)^4}{4!} + \ldots + \frac{(Ku^+)^N}{N!} + \ldots\right\} (N = 4, 5, 6, \ldots) \quad (4)$$

For $b = 3$, the eddy diffusivity profile recommended in reference [5] was:

$$\varepsilon_u^+ = 1 + \frac{K}{E}\left\{\frac{(Ku^+)^3}{3!} + \ldots + \frac{(Ku^+)^N}{N!} + \ldots\right\} (N = 3, 4, 5, \ldots) \quad (5)$$

In equations (4) and (5) K is a constant, usually taken to be about 0·4. E is another constant having a value of the order of magnitude of 10.

1.2.2.3 *Deissler's diffusivity profile.* In reference [6] Deissler defines a velocity profile by means of an integral equation

$$u^+ = \int_0^{y^+} \frac{\mathrm{d}y^+}{1 + n^2u^+y^+(1 - \exp[-n^2u^+y^+])} \quad (6)$$

where the recommended value of n is 0·124. This equation can be integrated to give a power series in u^+.

$$y^+ = u^+ + \frac{n^4(u^+)^5}{5} - \frac{n^6(u^+)^7}{14} + \tfrac{17}{270}\, n^8(u^+)^9 - \ldots. \quad (7)$$

Equation (7) can now be differentiated to give a diffusivity profile in the form of a power series in u^+ (since $\varepsilon_u^+ = \mathrm{d}y^+/\mathrm{d}u^+$). This expression is:

$$\varepsilon_u^+ = 1 + (nu^+)^4 - \tfrac{1}{2}(nu^+)^6 + \tfrac{17}{30}(nu^+)^8 - \ldots. \quad (8)$$

1.2.3 *The boundary conditions*

The boundary conditions on equation (1) for our problem are:

$$\left.\begin{aligned} \theta &= 0 \quad \text{at} \quad u^+ = 0 \quad \text{and} \quad \text{all } x^+ \\ \theta &= 1 \quad \text{at} \quad u^+ > 0 \quad \text{and} \quad x^+ = 0 \\ \theta &= 1 \quad \text{at} \quad u^+ = u_G^+ \quad \text{and} \quad x^+ \geqslant 0 \end{aligned}\right\} \quad (9)$$

We shall be particularly concerned with the prediction of local surface heat-transfer rates; these may be connected with the solution of $\theta(x^+, u^+)$ via the S-function [1, 7] which is defined by:

$$S \equiv \frac{N_{St}\, N_{Pr}}{\sqrt{(c_f/2)}} = -\left.\frac{\partial\theta}{\partial u^+}\right|_{u^+=0} \quad (10)$$

1.3 *Relation to Previous Work*

The solution of the problem considered in this paper has two asymptotes. The first is valid for small x^+ where all changes in the temperature profile are confined to the laminar sub-layer. These conditions reduce the problem to the one first solved by Levêque [8]. In terms of the S-function the Levêque solution is:

$$S = 0{\cdot}53835 \left(\frac{x^+}{N_{Pr}}\right)^{-1/3} \quad (11)$$

The second asymptote is valid for large values of x^+ and large Prandtl numbers. Deissler [6] gave a form of this solution which fitted his particular eddy diffusivity profile. In terms of the S-function, Deissler's solution is:

$$S = 0{\cdot}1116\, N_{Pr}^{1/4}. \quad (12)$$

This solution was generalized in reference [9], and gives

$$S = \frac{(a_b N_{Pr})^{1/b} \sin(\pi/b)}{\pi/b} \quad (13)$$

where a_b and b are from equation (2).

Gardner and Kestin [7] solved equation (1) on a digital computer for the constant wall temperature case using the Schmidt method of step-by-step integration. Their solutions were for the eddy diffusivity profile of equation (4) with values of K and E of 0·4 and 9·025 respectively. They presented numerical results for Prandtl numbers of 0·71, 1, 7, 30, 100 and 1000.

2. METHOD OF SOLUTION

2.1 *Definitions*

Let

$$\gamma \equiv a_b^{1/b}(u^+)^b \quad (14)$$

$$F_{b+k} \equiv a_{b+k}/[a_b^{(b+k)/b}] \qquad (k = 0, 1, 2, \ldots.) \quad (15)$$

i.e.

$$F_b = 1 \quad (16)$$

$$A \equiv \frac{N_{Pr}}{N_{Pr,t}} \quad (17)$$

$$Z \doteqdot \frac{x^+}{N_{Pr,t}} \quad (18)$$

2.2 *Modification of equation* (1)

With the definitions given above, equation (1) can be modified to

$$\left.\begin{aligned}\frac{\partial\theta}{\partial Z} = \frac{1}{\gamma}\Big\{&[1 + (A-2)\gamma^b \\ &+ F_{b+1}(A-2)\gamma^{b+1} + \ldots]\frac{\partial^2\theta}{\partial\gamma^2} \\ &+ [b(A-1)\gamma^{b-1} + F_{b+1}(b+1) \\ &\times (A-1)\gamma^b + \ldots]\frac{\partial\theta}{\partial\gamma}\Big\}\end{aligned}\right\} \quad (19)$$

Let us now suppose that θ can be expressed in the form of a series,

$$\theta \equiv \theta_0 + \sum_{n>0} \theta_n Z^n \quad (20)$$

where the θ_n's are all functions of v only, where

$$v \equiv \gamma Z^{-1/3} \quad (21)$$

then equation (19) becomes a total differential equation:

$$\left.\begin{aligned}v\sum_n n\,\theta_n Z^n = &[1 + (A-2)v^b Z^{b/3} \\ &+ F_{b+1}(A-2)v^{b+1}Z^{(b+1)/3} + \ldots] \\ &\times [\theta_0'' + \sum_n \theta_n'' Z^n] + \left[\frac{v^2}{3} + b(A-1)\right. \\ &\times v^{b-1}Z^{b/3} + F_{b+1}(b+1)(A-1) \\ &\times \left. v^b Z^{(b+1)/3} + \ldots\right][\theta_0' + \sum_n \theta_n' Z^n]\end{aligned}\right\} \quad (22)$$

The prime in this equation stands for differentiation with respect to v.

Equation (22) can be solved by the method of equating coefficients.

The boundary conditions for equation (22) are:

$$\left.\begin{aligned}\theta &= 0 \quad \text{at} \quad v = 0 \\ \theta &\to 1 \quad \text{as} \quad v \to \infty\end{aligned}\right\} \quad (23)$$

2.2.1 *Coefficient of* Z^0. For this term, equation (22) becomes:

$$\theta_0'' + \frac{v^2}{3}\theta_0' = 0. \quad (24)$$

This equation can be solved exactly, and gives

$$\theta_0 = C\int_0^v [\exp\{-v^3/9\}]\,dv + B. \quad (25)$$

From the boundary conditions (23), B is zero, and

$$C = \frac{1}{\int_0^\infty \exp[-v^3/9]\,dv} = 0{\cdot}53835. \quad (26)$$

Therefore,

$$\theta_0' = 0{\cdot}53835\exp[-v^3/9]. \quad (27)$$

This is the Levêque solution referred to above.

2.2.2 *Other coefficients.* The boundary conditions on θ_n, for non-zero values of n, become

$$\left.\begin{aligned}\theta_n &= 0 \quad \text{at} \quad v = 0 \\ \theta_n &\to 0 \quad \text{as} \quad v \to \infty\end{aligned}\right\} \quad (28)$$

It can be shown that the only values of n for which θ_n and all of its derivatives are not zero, are

$$n = 0, \frac{b}{3}, \ldots, \frac{b+k}{3}, \ldots \quad (k = 0, 1, 2, 3, \ldots) \quad (29)$$

Thus, from equation (19), the series expansion for θ becomes

$$\theta = \theta_0 + \theta_{b/3}Z^{b/3} + \ldots + \theta_{(b+k)/3}Z^{(b+k)/3} + \ldots \quad (30)$$

and from equation (9)

$$S = \theta_0'\left.\frac{dv}{du^+}\right|_{u^+=0} + \theta_{b/3}'Z^{b/3}\left.\frac{dv}{du^+}\right|_{u^+=0} + \ldots + \theta_{(b+k)/3}'Z^{(b+k)/3}\left.\frac{dv}{du^+}\right|_{u^+=0} + \ldots. \quad (31)$$

The details of the method used to determine the values of $\theta_{(b+k)/3}'$ are given in the Appendix.

3. RESULTS

Equations in the form of equations (I-10) and (I-11) were solved on a digital computer using step-by-step integration and the results were checked by repeating the calculations with halved steps.

3.1 *Case where* $b = 4$ *in Equation* (2)

For this case the first eight terms of the series solution for $S(x^+, N_{Pr}, N_{Pr,t})$ were found to be:

$$S = 0{\cdot}53835\left(\frac{x^+}{N_{Pr}}\right)^{-1/3} + a_4[0{\cdot}65454\,A - 0{\cdot}10909]\left(\frac{x^+}{N_{Pr}}\right)$$
$$+ a_4^{5/4}F_5[1{\cdot}03005\,A - 0{\cdot}14715]\left(\frac{x^+}{N_{Pr}}\right)^{4/3} + (a_4)^{3/2}F_6[1{\cdot}7227\,A - 0{\cdot}2153]\left(\frac{x^+}{N_{Pr}}\right)^{5/3}$$
$$- a_4^2[0{\cdot}799\,(A-2)^2 + 1{\cdot}514\,(A-2) + 0{\cdot}244\,(A-1) + 0{\cdot}50186\,\{F_8(A-2) - (2A-3)\}$$
$$- 10{\cdot}8948\,(F_8 - 1)(A-1)]\left(\frac{x^+}{N_{Pr}}\right)^{7/3} - (a_4)^{9/4}(2{\cdot}7247)\,A^2F_5\left(\frac{x^+}{N_{Pr}}\right)^{8/3}$$
$$- (a_4)^{5/2}(0{\cdot}75457)\,A^2F_6\left(\frac{x^+}{N_{Pr}}\right)^3 + a_4^3(2{\cdot}83)\,A^3\left(\frac{x^+}{N_{Pr}}\right)^{11/3} + \text{higher order terms} \quad (32)$$

where the values of a_4, F_5, F_6, and F_8 [defined by equations (2), (15) and (16)] are dependent upon the particular eddy diffusivity profile used. The halved-step check indicated that the $0{\cdot}244\,(A - 1)$ term contained in the coefficient of the $(x^+/N_{Pr})^{7/3}$ term of equation (32) was only accurate to two per cent. Since this term is small for large A (i.e. large $N_{Pr}/N_{Pr,t}$), the calculations were continued giving the coefficients for (x^+/N_{Pr}) to the 8/3, 9/3, and 11/3 powers. These coefficients are strictly valid for large Prandtl numbers only.

3.1.1 *Values of the constants in equation* (32) *for various diffusivity profiles*

3.1.1.1 *Diffusivity profile of equation* (4). Gardner and Kestin [7] used the diffusivity profile of equation (4) with values of K and E equal to 0·4 and 9·025 respectively; for comparison, the same values will be used here. These, with equations (2), (15) and (16), imply:

$$b = 4;\, a_4 = 4{\cdot}73 \times 10^{-5};\, F_5 = 0{\cdot}965;\quad F_6 = 0{\cdot}776;\, F_8 = 0{\cdot}322 \quad (33)$$

The S-function obtained by using these constants in equation (32) is shown in Fig. 1* for several values of Prandtl number. The results given by Gardner and Kestin for these Prandtl numbers are also shown in Fig. 1. At large values of x^+ the series of equation (32) diverges from the Gardner–Kestin results. This divergence occurs because equation (32) is actually an infinite series, and the higher-order-terms [which have not been determined for equation (4)] become important at these large values of x^+. Up to the point of divergence, agreement between the two curves is excellent, and the doubts expressed in reference [9] about the accuracy of the Gardner–Kestin results at high Prandtl numbers are apparently not justified.

* In all of the figures the turbulent Prandtl number was taken as unity. To use a different constant for the turbulent Prandtl number it is necessary to replace N_{Pr}, S and x^+, in these figures, with

$$\frac{N_{Pr}}{N_{Pr,t}},\ \frac{S}{N_{Pr,t}} \text{ and } \frac{x^+}{N_{Pr,t}}$$

3.1.1.2 *Diffusivity profile of equation* (8). For this diffusivity profile, the values of the constants in equation (32) are:

$$b = 4;\quad a_4 = 2{\cdot}36 \times 10^{-4};\quad F_5 = F_7 = 0;\quad F_6 = -0{\cdot}5;\quad F_8 = 0{\cdot}567 \quad (34)$$

The curve obtained from equation (32) for these constants is also shown in Fig. 1. Since the value of a_4 above for Deissler's diffusivity profile is larger than the value of Spalding's a_4 (for a b of 4) given in (33), the curves given by equation (32) for Deissler's profile might be expected to deviate from the Levêque solution sooner than the curve for Spalding's profile. This is seen to be the case in Fig. 1. This higher value of a_4 for Deissler's profile also reduces the maximum value of x^+ for which equation (32) is valid.

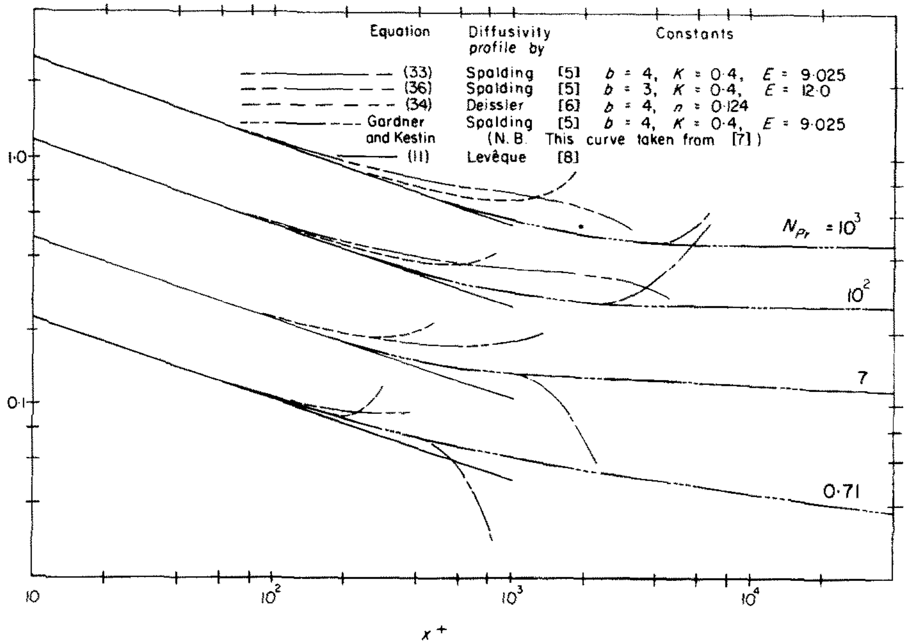

FIG. 1. S vs. x^+ from equations (33), (34) and (36) compared with the Gardner–Kestin results for Prandtl numbers of 0·71, 7, 100 and 1000.

3.2 *Case where $b = 3$ in Equation* (2)

For this case the first five terms in the series expansion of S were found to be:

$$S = 0{\cdot}53835\left(\frac{x^+}{N_{Pr}}\right)^{-1/3} + a_3[0{\cdot}4486\,A - 0{\cdot}0897]\left(\frac{x^+}{N_{Pr}}\right)^{2/3} + (a_3)^{4/3}$$
$$\times\, F_4[0{\cdot}65454\,A - 0{\cdot}10909]\left(\frac{x^+}{N_{Pr}}\right) + (a_3)^{5/3}F_5[1{\cdot}03005\,A - 0{\cdot}14715]\left(\frac{x^+}{N_{Pr}}\right)^{4/3}$$
$$- a_3^2[0{\cdot}27315\,(A-2)^2 + 0{\cdot}53108\,(A-2) - 0{\cdot}011162\,(A-1) - 1{\cdot}50740\,(F_6 + A - 2)$$
$$- 0{\cdot}17227\,(\mathrm{F}_6 - 1)(A-1)]\left(\frac{x^+}{N_{Pr}}\right)^{5/3} - \text{higher order terms} \quad (35)$$

where, again, the values of a_3, F_4, F_5 and F_6, are dependent upon the particular eddy diffusivity profile used. The calculations were stopped at this point since the half-step check indicated the coefficient of the next term would be inaccurate.

The diffusivity profile of equation (5) has a value of b equal to three. For this diffusivity profile, with values of K and E equal to 0·4 and 12 respectively, the values of the constants in equation (35) are:

$$b = 3;\quad a_3 = 3{\cdot}56 \times 10^{-4};\quad F_4 = 1{\cdot}42;$$
$$F_5 = 1{\cdot}54;\quad F_6 = 1{\cdot}50 \quad (36)$$

The curve obtained from equation (35) for these constants is also shown in Fig. 1. The main difference in the first few terms of equations (32) and (35) is the addition of an $(x^+/N_{Pr})^{2/3}$ term in equation (35). This additional term caused the S-function to deviate from the Levêque solution at smaller values of x^+ for Spalding's eddy diffusivity profile with a b of three than for his profile with a b of four. This effect is seen in Fig. 1. It is interesting to note that Deissler's profile [equation (8)], with a b of four, stays fairly close to the $b = 3$ profile. In fact, for a Prandtl number of seven, they are nearly identical.

3.3 *Discussion and Comparison with Experiments*

Van Shaw [10] has obtained experimental values of $\bar{S}$ for the analogous problem of electrochemical mass transfer in a tube. He presented data for a large range of Reynolds numbers ($N_{Re,d}$) and tube lengths for a fluid with a Schmidt number of 2400. These results are shown in Fig. 2. The $\bar{S}$ curves shown on the figure were obtained, for the diffusivity profiles used in this study [equations (33), (34) and (36)], from the relationship

$$\bar{S} = \frac{1}{L^+} \int_0^{L^+} S \, \mathrm{d}x^+ \tag{37}$$

The right-hand asymptote for these curves has been estimated from equation (13), and the intermediate region has been drawn freehand.

Recently Schütz [11] obtained local experimental values of S for a fluid with a Schmidt number of 2170. Like Van Shaw, Schütz also used the electrochemical technique in a tube and obtained results for $N_{Re,d}$ between 5000 and 50000; however, data for $N_{Re,d}$ of less than 20000 have been omitted from Fig. 3 since experimental evidence (e.g. reference [12]) indicates that the velocity profiles are dependent upon $N_{Re,d}$ for $N_{Re,d}$ of less than 20000. Curves of S have been plotted from equations (33), (34) and (36) for low x^+ values; the asymptotic values have again been estimated from equation (13); and the intermediate region has been drawn freehand.

In Fig. 2 it is seen that Van Shaw's experimental data fall below the Levêque solution in the low L^+ region, and in Fig. 3 Schütz's data are above the Levêque solution in this region. These small discrepancies could be caused by many things (e.g. electrode edge effects, small errors in determining the Schmidt number, etc.) and are indicative of the difficulty encountered in

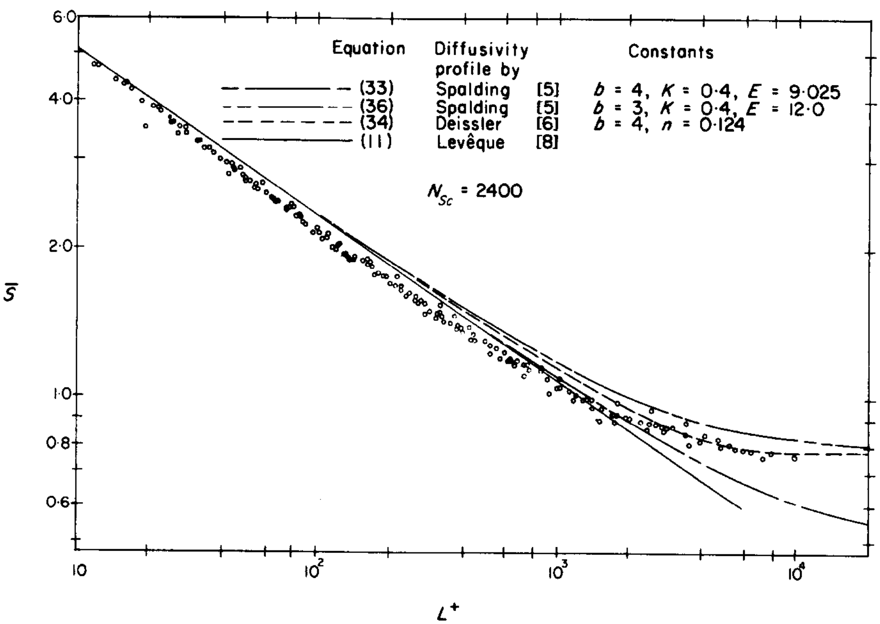

FIG. 2. $\bar{S}$ vs. L^+ from Van Shaw's experimental data [10] and theory.

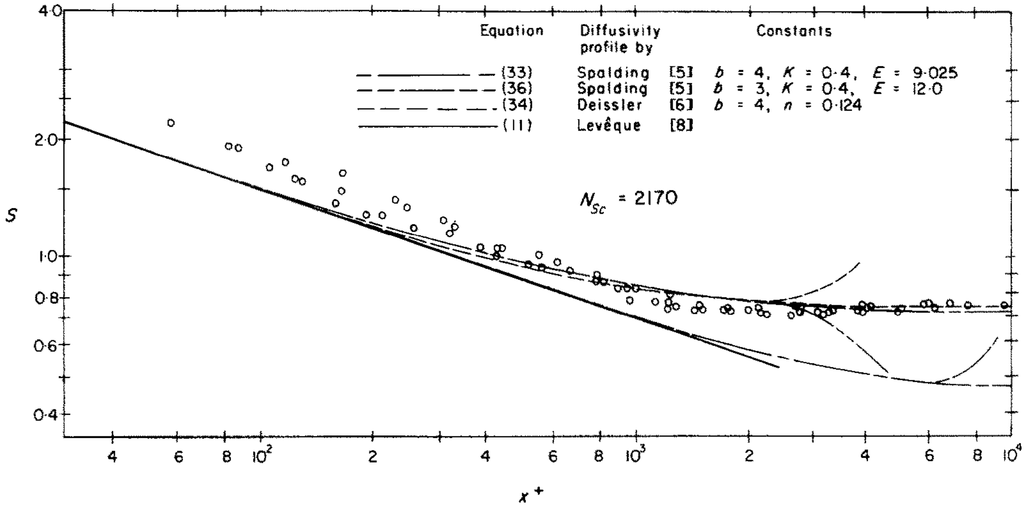

FIG. 3. S vs. x^+ from Schütz's experimental data [11] and theory.

measuring mass-transfer (or heat-transfer) rates at low values of x^+ (or L^+). Even though it is likely that small errors exist in the experimental data, it is evident from Figs. 2 and 3 that Spalding's diffusivity profile with a b of four [equation (4)]—the lower chain-dotted curve in both figures—does not fit the experimental data. The other two curves, representing Spalding's profile for a b of three [equation (5)] and Deissler's profile [equation (8)] both fit the experimental data reasonably well.

4. SUMMARY

An analytical expression has been found for the local heat-transfer to a steady, uniform-property turbulent boundary layer from the initial portion of an isothermal section of a smooth flat plate downstream of a step-wise discontinuity in wall temperature. These results are expressed in the form of $S(N_{Pr}, x^+)$ and are valid for cases where the diffusivity profiles can be expressed in the form of a power series in u^+.

In reference [9] the accuracy of the Gardner–Kestin results was questioned. The results of the present study were compared with those of Gardner and Kestin and excellent agreement was obtained, indicating that the doubts expressed in reference [9] were not justified.

Published mass-transfer data (for fluids with Schmidt numbers greater than 2000) were used to determine the best value of the coefficient a_b for the eddy diffusivity profile in equation (2). For a value of b of four, Deissler's diffusivity profile ($a_4 = 2{\cdot}36 \times 10^{-4}$) fits the data quite well; correspondingly, the values of S obtained using the diffusivity profile in equation (4) ($a_4 = 4{\cdot}73 \times 10^{-5}$) are low at high x^+. Equation (5) gives a good fit to the data if values of K and E of 0·4 and 12 are used (i.e. $a_3 = 3{\cdot}56 \times 10^{-4}$).

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APPENDIX

Determination of the Individual Values of θ'_n

Consider the case where b in equation (2) is four. By equating coefficients of Z^n in equation (22) the following total differential equations are obtained.

$$\theta''_{4/3} + \frac{v^2}{3}\theta'_{4/3} - \frac{4}{3}v\theta_{4/3} + (A-2)\,v^4\theta''_0 + 4(A-1)\,v^3\theta'_0 = 0 \tag{I-1}$$

$$\theta''_{5/3} + \frac{v^2}{3}\theta'_{5/3} - \frac{5}{3}v\theta_{5/3} + F_5(A-2)\,v^5\theta''_0 + 5F_5(A-1)\,v^4\theta'_0 = 0 \tag{I-2}$$

$$\theta''_2 + \frac{v^2}{3}\theta'_2 - 2v\theta_2 + F_6(A-2)\,v^6\theta''_0 + 6F_6(A-1)\,v^5\theta'_0 = 0 \tag{I-3}$$

$$\theta''_{7/3} + \frac{v^2}{3}\theta'_{7/3} - \frac{7}{3}v\theta_{7/3} + F_7(A-2)\,v^7\theta''_0 + 7F_7(A-1)\,v^6\theta'_0 = 0 \tag{I-4}$$

$$\theta''_{8/3} + \frac{v^2}{3}\theta'_{8/3} - \frac{8}{3}v\theta_{8/3} + (A-2)\,v^4\theta''_{4/3} + 4(A-1)\,v^3\theta'_{4/3} + [F_8(A-2) - (2A-3)]\,v^8\theta''_0 + 8(A-1)(F_8 - 1{\cdot}5)\,v^7\theta'_0 = 0 \tag{I-5}$$

For large Prandtl numbers,

$$\theta''_3 + \frac{v^2}{3}\theta'_3 - 3v\theta_3 + v^4\theta''_{5/3} + 4v^3\theta'_{5/3} + F_5v^5\theta''_{4/3} + 5F_5v^4\theta'_{4/3} = 0 \tag{I-6}$$

$$\theta''_{10/3} + \frac{v^2}{3}\theta'_{10/3} - \frac{10}{3}v\theta_{10/3} + v^4\theta''_2 + 4v^3\theta'_2 + F_5v^5\theta''_{5/3} + 5F_5v^4\theta'_{5/3} + F_6v^6\theta''_{4/3} + 6F_6v^5\theta'_{4/3} = 0 \tag{I-7}$$

$$\theta''_{11/3} + \frac{v^2}{3}\theta'_{11/3} - \frac{11}{3}v\theta_{11/3} + v^4\theta''_{7/3} + 4v^3\theta'_{7/3} + F_5v^5\theta''_2 + 5F_5v^4\theta'_2 + F_6v^6\theta''_{5/3} + 6F_6v^5\theta'_{5/3} + F_7v^7\theta''_{4/3} + 7F_7v^6\theta'_{4/3} = 0 \tag{I-8}$$

$$\theta''_4 + \frac{v^2}{3}\theta'_4 - 4v\theta_4 + v^4\theta''_{8/3} + 4v^3\theta'_{8/3} = 0 \tag{I-9}$$

where the θ_n terms in equations (I-1)–(I-9) are functions of v and Prandtl number. By superposition it is possible to obtain a new set of total differential equations which are independent of Prandtl number. Consider

$$\varphi''_{4/3,1} + \frac{v^2}{3}\varphi'_{4/3,1} - \frac{4}{3}v\varphi_{4/3,1} + v^4\theta''_0 = 0 \tag{I-10}$$

$$\varphi''_{4/3,2} + \frac{v^2}{3}\varphi'_{4/3,2} - \frac{4}{3}v\varphi_{4/3,2} + 4v^3\theta'_0 = 0 \tag{I-11}$$

then

$$\theta_{4/3} = (A-2)\,\varphi_{4/3,1} + (A-1)\,\varphi_{4/3,2} \tag{I-12}$$

and

$$\theta'_{4/3} = (A-2)\,\varphi'_{4/3,1} + (A-1)\,\varphi'_{4/3,2} \tag{I-13}$$

where the φ terms are functions of v only. The values for the φ' terms at the wall ($u^+ = 0$) can be determined by solving equations (I-10) and (I-11) using finite difference techniques. Then the values of θ'_n at the wall can be obtained as a function of Prandtl number only.

Equations (I-2)–(I-9) can be treated the same way giving equations similar to (although more complex than) equations (I-10)–(I-13).

Résumé—On a trouvé une solution sous forme de série de l'équation aux dérivées partielles paraboliques à laquelle obéit le transport de chaleur à partir d'une paroi lisse vers une couche limite turbulente permanente et à propriétés physiques constantes. La solution est valable pour la partie initiale de la paroi immédiatement en aval d'une discontinuité en échelon de température pariétale. L'équation aux dérivées partielles est celle de la température dans fluide dont le nombre de Prandtl est plus grand que 0,5, lorsqu'un profil de vitesse universel existe partout où le gradient de température est appréciable. Pour obtenir une solution de l'équation sous forme de série, les diffusivités thermique et cinématique ont été écrites chacune sous la forme d'une série en puissances de u^+. Une expression générale a été utilisée pour la série en puissances qui peut être modifiée facilement afin de s'adapter à plusieurs profils de diffusivité conseillés dans la littérature. Comme exemple de cette souplesse d'emploi, des calculs ont été effectués (pour plusieurs nombres de Prandtl) avec trois profils de diffusivité courants. Quelques coefficients de u^+, dans la série en puissances de la diffusivité ont été obtenus par comparaison des solutions de l'équation aux dérivées partielles avec les résultats expérimentaux publiés récemment.

Zusammenfassung—Für die parabolische partielle Differentialgleichung des Wärmeüberganges von einer glatten Wand an eine stationäre turbulente Grenzschicht gleichmässiger Stoffeigenschaften wurde eine Reihenlösung gefunden. Diese Lösung gilt für den Anfangsteil der Wand unmittelbar stromabwärts einer stufenförmigen Diskontinuität der Wandtemperatur. Die Differentialgleichung gibt die Temperaturverteilung wieder in einer Flüssigkeit mit einer Prandtl-Zahl grösser als 0,5, soweit bei merklichem Temperaturgradient in der Flüssigkeit ein universelles Geschwindigkeitsprofil vorhanden ist. Um eine Reihenlösung der Differentialgleichung zu erhalten wurden die Koeffizienten des turbulenten Energie- und Impulsaustausches in Form einer Potenzreihe in u^+ ausgedrückt. Für die Potenzreihe wurde eine allgemeine Form gewählt um sie zur Anpassung an viele in der Literatur empfohlene Austauschprofile modifizieren zu können. Als Beispiel für die Flexibilität wurden Berechnungen (für verschiedene Prandtl-Zahlen) mit drei oft verwendeten Austauschprofilen durchgeführt. Einige der Koeffizienten von u^+ in der Austauschpotenzreihe wurden aus Vergleichen der Lösungen der Differentialgleichungen mit kürzlich veröffentlichten Versuchswerten abgeleitet.

Аннотация—Для параболического дифференциального уравнения в частных производных, описывающего перенос тепла от гладкой стенки к стационарному турбулентному пограничному слою с однородными свойствами, найдено решение методом разложения в ряд. Решение справедливо для начального участка стенки вдоль по потоку, начиная с точки ступенчатого разрыва температуры стенки. Дифференциальное уравнение описывает распределение температуры в жидкости ($Pr > 0{,}5$) при наличии в последней универсального профиля скорости там, где имеется заметный градиент температуры. Для получения решения дифференциального уравнения с помощью разложения в ряд коэффициенты диффузии тепла и количества движения представлены в виде степенного ряда по u^+. Был взят общий вид степенного ряда, который можно легко преобразовать для описания многих профилей коэффициентов диффузии, рекомендуемых в литературе. В качестве примера проведены расчёты (при разных значениях чисел Прандтля) для трёх обычно используемых профилей диффузии. Некоторые коэффициенты u^+ в степенном диффузионном ряду выведены с помощью сравнения решений дифференциального уравнения с недавно опубликованными экспериментальными данными.